Foundations of Analysis

[DRAFT]

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Mathsheetz

Archimedian Principle (AP)

For every real number $x \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.

Equivalently, for every real number $x > 0$, there exists a natural number n such that $\frac{1}{n} < x$.

Preliminary Definitions

Let A and B be any two sets. Then

$$x \in A \cup B \iff x \in A \textit{ or } x \in B$$

$$x \in A \cap B \iff x \in A \textit{ and } x \in B$$

$$A \subset B \iff x \in A \to x \in B$$

If $\{A_i\}$ is a countable collection of sets, we have the following:

$$x \in \bigcup_{i=1}^{\infty} A_i \iff x \in A_i \textit{ for some } i$$

$$x \in \bigcap_{i=1}^{\infty} A_i \iff x \in A_i \textit{ for all } i$$

Let $E \subset \mathbb{R}$ be any set.

M is an **upper bound** of E if, for all $x \in E$, $x \leq M$. m is a **lower bound** of E if, for all $x \in E$, $x \geq m$.

The **supremum** of a set E is defined to be the smallest upper bound of the set. That is, $\sup(E)$ must satisfy the following:

$$1.\ x \in E \implies x \leq \sup(E)$$

$$2.\ M \text{ is an upper bound of } E \implies \sup(E) \leq M$$

The **infimum** of a set E is defined to be the greatest lower bound of the set. That is, $\inf(E)$ must satisfy the following:

$$1.\ x \in E \implies x \geq \inf(E)$$

$$2.\ m \text{ is a lower bound of } E \implies \inf(E) \geq m$$

Note that if an upper bound of E belongs to the set E, it is automatically the supremum. Similarly, if a lower bound of E belongs to the set E, it is automatically the infimum. E.g. $1 \in [0, 1]$ so $\sup[0, 1] = 1$.

Theorem (Nested Interval Theorem)

The intersection of a sequence of nested, closed, bounded (nonempty) intervals $I_1 \supset I_2 \supset I_3 \supset ...$ is nonempty, that is,

$$\cap_{n=1}^{\infty} I_n \neq \emptyset$$

Examples.

$$\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] = \{0\}$$

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset$$

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1 - \frac{1}{n}\right) = (0, 1)$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1]$$

$$\sup(0, 1) = 1 \quad \sup\big((0, 1) \cap \mathbb{Q}\big) = 1 \quad \inf(0, 1) = 0 \quad \inf\big((0, 1) \cap \mathbb{Q}\big) = 0$$

Triangle Inequality

Let x, y be real numbers. Then we have:

$$|x+y| \leq |x| + |y|$$
$$|x-y| \geq ||x| - |y||$$

Note that this can be written

$$||x| - |y|| \leq |x+y| \leq |x| + |y|$$
$$||x| - |y|| \leq |x-y| \leq |x| + |y|$$

Limits

The sequence $\{a_n\}$ **converges** to a real number a if, for every $\epsilon > 0$, there exists an $N \in N$ such that

$$|a_n - a| < \epsilon \quad \text{for all } n \geq N$$

We write $\lim_{n\to\infty} a_n = a$ when this definition holds.

The sequence $\{a_n\}$ is **cauchy** if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \quad \text{for all } n, m \geq N$$

Theorem: (Cauchy $\Longleftrightarrow$ Convergent)

A sequence $\{a_n\}$ *in the real numbers* $\mathbb{R}$ converges to some limit $a \in \mathbb{R}$ if and only if it is Cauchy.

Important examples:

The sequence $\{1, 1.4, 1.41, 1.414, 1.4142, ...\}$ of rational numbers is cauchy in both $\mathbb{R}$ and $\mathbb{Q}$. The sequence converges to $\sqrt{2}$ in $\mathbb{R}$, but does not converge in $\mathbb{Q}$ (since $\sqrt{2}$ is not rational!)

The sequence $\{1/n\}$ is cauchy and converges to 0.

The sequence of *partial sums* $\{\sum_{k=1}^{n} \frac{1}{k}\}$, also written $\{S_n\}$ where $S_n = \sum_{k=1}^{n} \frac{1}{k}$, and looks like

$$\{1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{3},\ 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}...\}$$

is NOT cauchy and diverges to infinity (write the proof). Note that $|S_{n+1} - S_n| = \frac{1}{n+1} \to 0$, but this not strong enough for the sequence of partial sums to be cauchy (hence the need for n and m in the cauchy definition).

The sequence of *partial sums* $\{\sum_{k=0}^{n} \frac{1}{2^k}\}$, which looks like

$$\{1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{4},\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, ...\}$$

IS cauchy and converges to 2 (write the proof).

Squeeze Theorem.

If, for all n, we can squeeze the sequence b_n between two sequences a_n and c_n, that is,

$$a_n \leq b_n \leq c_n \quad \text{for all } n \in \mathbb{N}$$

Then

$$\lim a_n \leq \lim b_n \leq \lim c_n$$

More generally, limits preserve *NONSTRICT* inequalities. That is,

$$a_n \leq b_n \Longrightarrow \lim a_n \leq \lim b_n$$

$$a_n < b_n \Longrightarrow \lim a_n \leq \lim b_n$$

Example: The sequence $\{\frac{1}{n}\sin(n)\}$ can be squeezed between $-1/n$ and $1/n$, that is,

$$-\frac{1}{n} \leq \frac{1}{n}\sin(n) \leq \frac{1}{n}$$

So it follows that

$$0 = \lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{1}{n}\sin(n) \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

By squeeze theorem, $\lim_{n\to\infty} \frac{1}{n}\sin(n) = 0$.

Example: (Trick) Consider the sequence $\{\sqrt{n+1} - \sqrt{n}\}$. Does the sequence converge?

Answer: Yes. Use the trick

$$\lim(\sqrt{n+1} - \sqrt{n}) = \lim \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \lim \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Now use squeeze theorem

$$0 \leq \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}} \to 0$$

To see rigorously that $\frac{1}{\sqrt{n}} \to 0$, use the limit definition. Fix $\epsilon > 0$. By (AP) there exists N such that $N > \frac{1}{\epsilon^2}$. For all $n \geq N$,

$$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} \leq \epsilon$$

Since $N > 1/\epsilon^2$ implies $1/\sqrt{N} < \epsilon$. Make sure you understand every step of this example.

Weierstrass Theorem (!!)

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Limits at ∞

We say $\{a_n\} \to \infty$, or $\{a_n\}$ diverges to ∞, if for every $M > 0$ there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$, $a_n > M$.

Continuity

A function $f(x)$ is **continuous at a point** $x = a$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

This is equivalent to requiring

$$\lim_{x \to a} f(x) = f(a)$$

The function is said to be **continuous** if it is continuous at all points in its domain.

A function $f(x)$ is **uniformly continuous** on its domain D if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\forall\, a \in D,\ |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

The choice of δ does NOT depend on the choice of the point a, which is the key difference between uniform continuity and pointwise continuity.

Theorem (Continuous on Closed, Bounded Interval $\implies$ Unif Cts)

A function which is continuous on a closed, bounded interval $[a, b]$ is uniformly continuous on $[a, b]$.

Theorem (Cts Extension $\iff$ Uniform Cts)

A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) if and only if it admits a continuous extension to $[a, b]$, that is, if and only if the limits

$$\lim_{x \to a^+} f(x), \quad \lim_{x \to b^-} f(x)$$

are well defined. Note that the values of these limits will be exactly $f(a)$ and $f(b)$ of the extension.

Key Examples:

1. The function x^2 is uniformly continuous on $(0, 1)$, but NOT uniformly continuous on $[1, \infty)$.

Proof: x^2 continuously extends to $[0, 1]$ so it is uniformly continuous on $(0, 1)$. To see that it is NOT uniformly continuous on $[1, \infty)$, fix $\epsilon = 1/2$ and suppose for sake of contradiction there exists a $\delta > 0$ such that, for all $a \in [1, \infty)$,

$$|x - a| < \delta \implies |x^2 - a^2| < \frac{1}{2}$$

Since $|x^2 - a^2| = |x + a||x - a|$, fix $x = a + \frac{\delta}{2}$, so that certainly $|x - a| = \frac{\delta}{2} < \delta$, but

$$|x^2 - a^2| = |x + a||x - a| = |2a + \frac{\delta}{2}||\frac{\delta}{2}|$$

since $a \in [1, \infty)$ can be chosen arbitrarily large, set $a = \max\{1, \frac{1}{\delta}\}$ so that

$$|x^2 - a^2| = |2a + \frac{\delta}{2}||\frac{\delta}{2}| > |2a||\frac{\delta}{2}| = a\delta \geq 1$$

Since the uniform continuity definition requires that $|x^2 - a^2| < \frac{1}{2}$, this is a contradiction.

2. The function $\sin(1/x)$ is continuous on $(0, 1)$, but NOT uniformly continuous (since it does not admit a continuous extension at 0, as $\lim_{x \to 0} \sin(1/x)$ DNE)

3. The function $x \sin(1/x)$ is uniformly continuous on $(0, 1)$ since it admits a continuous extension to $[0, 1]$.

Theorem: (Unif Cts on Bounded Interval $\implies$ Bounded)

Any uniformly continuous function on a bounded interval (a, b) is bounded.

Example. $1/x$ is not bounded on $(0,1)$ and hence not uniformly continuous.

Theorem: (Cts on Closed, Bounded Interval $\implies$ Bounded, Assumes Extrema)

Any continuous function on a closed and bounded interval $[a,b]$ is bounded, and moreover assumes its maxima and minima. That is, there exists at least one point $c \in [a,b]$ such that

$$\sup_{x\in[a,b]} f(x) = f(c)$$

and likewise at least one point $d \in [a,b]$ such that $\inf_{x\in[a,b]} f(x) = f(d)$.

****Intermediate Value Theorem (IVT)****

Let f be a continuous function on the closed bounded interval $[a,b]$. Choose any y such that

$$\inf_{[a,b]} f(x) \le y \le \sup_{[a,b]} f(x)$$

Then there exists $c \in [a,b]$ such that $f(c) = y$.

Key example: Let f, g be continuous functions on $[a,b]$ with $f(a) < g(a)$ and $f(b) > g(b)$. Show that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

Proof. Apply MVT to the function $f - g$. Since $f(a) - g(a) < 0$ and $f(b) - g(b) > 0$, we can choose $y = 0$ in the statement of the MVT, so there must exist $c \in [a,b]$ such that $f(c) - g(c) = 0$.

Example: The function x^3 is continuous on $[0,10]$ with minimum 0 and maximum 1000. Consider the number 777. Since $0 < 777 < 1000$, there must exist $c \in [0,10]$ such that $c^3 = 777$, that is, c is the cube root of 777.

Sequences of Functions

Similar to sequences of numbers, we can consider sequences of functions $\{f_n(x)\}$. For each fixed value of x, we get a sequence of numbers.

The sequence $\{f_n\}$ **converges pointwise** if, for all fixed $x \in \mathbb{R}$, the sequence of numbers $\{f_n(x)\}$ converges in the traditional sense to a limit point $f(x) \in \mathbb{R}$.

That is, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have $|f_n(x) - f(x)| < \epsilon$. We write the pointwise limit as

$$\lim_{n\to\infty} f_n(x) = f(x)$$

.

The sequence $\{f_n\}$ **converges uniformly** to a function f if,

for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, and *for all* $x \in D$, we have $|f_n(x) - f(x)| \le \epsilon$

Notice the key difference: For the uniform definition, the choice of N does not depend on the choice of the point x.

Uniform Limit Theorem: (Uniform Limit of Cts Functions is Cts)

If a sequence of continuous functions $\{f_n\}$ *converges uniformly* to f, then f must be continuous.

Key example. The sequence of functions

$$\{f_n(x)\} = \{\frac{nx}{1+nx}\}$$

converges pointwise to 1 when $x \neq 0$ and 0 when $x = 0$. Since the pointwise limit is discontinuous, this implies that $\{f_n\}$ does NOT converge uniformly.

Theorem (Uniform Convergence Test) Let $\{f_n(x)\}$ be a sequence of functions with domain D. If there is a sequence of numbers b_n with

$$|f_n(x)| \leq b_n \text{ for all } n$$

then

$$b_n \to 0 \text{ implies } \{f_n\} \to 0 \text{ uniformly.}$$

Note: This theorem is extremely useful in more general cases, due to the fact that $\{f_n\} \to f$ uniformly if and only if $\{f_n - f\} \to 0$ uniformly.

Differentiability

A continuous function f is **differentiable** at the point $x \in \mathbb{R}$ if the below limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{*}$$

If the limit exists, we define $f'(x)$ to be the value of the limit. An equivalent definition is

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

Theorem: (Diffble $\Longrightarrow$ Cts)

Let f be a differentiable function on an interval $[a, b]$. That is, at each point $x \in [a, b]$ the above limit (*) exists. Then f is continuous.

Example. The function $f(x) = x^2 \sin(1/x)$ is differentiable on $(0, 1)$ and uniformly continuous on $(0, 1)$, in particular it admits a continuous extension to 0. The derivative also exists on $[0, 1]$, and in particular $f'(0) = 0$, but $f'(x)$ is not continuous on $[0, 1]$. Therefore if a function is differentiable, the derivative need not be continuous.

**** Mean Value Theorem (MVT) ****

Let f be a differentiable function on an interval $[a, b]$. Then there exists $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value Theorem Let f, g be differentiable on $[a, b]$. Assume further that $g' \neq 0$. Then there exists $c \in [a, b]$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Consequences of MVT:
1. If the derivative of a function is everywhere 0, the function must be constant.
2. If two functions have the same derivative, they must differ only by a constant.
3. The antiderivative of a function is unique up to addition by a constant.

Theorem: (Bounded Derivative $\Longrightarrow$ Unif Cts)

Let f be a differentiable function on (a, b) such that f' is bounded. Then f is uniformly continuous on (a, b).

Monotonicity

A function f is **monotone increasing** if

$$a < b \Longrightarrow f(a) \leq f(b)$$

and **strictly monotone increasing** if

$$a < b \Longrightarrow f(a) < f(b)$$

Similarly, a function f is **monotone decreasing** if

$$a < b \Longrightarrow f(a) \geq f(b)$$

and **strictly monotone decreasing** if

$$a < b \Longrightarrow f(a) > f(b)$$

The sign of the derivative characterizes monotonicity, that is

$$f \text{ monotone increasing} \Longleftrightarrow f' \geq 0$$

$$f \text{ strictly monotone increasing} \iff f' > 0$$
$$f \text{ monotone decreasing} \iff f' \leq 0$$
$$f \text{ strictly monotone decreasing} \iff f' < 0$$

Inverse Function Theorem

If f is continuous and strictly monotone (increasing or decreasing) on an open interval I, with nonzero derivative $f'(a) \neq 0$ at a point a, then the inverse function f^{-1} is differentiable at $f(a)$ and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

Example. Consider the function $f(x) = x^2$ on $[0, 10]$ which has inverse $f^{-1}(x) = \sqrt{x}$. Notice that $f(3) = 9$ and $f'(3) = 6$, so by the above theorem

$$(f^{-1})'(f(3)) = (\sqrt{x})'(9) = \frac{1}{f'(3)} = \frac{1}{6}$$

Indeed, the derivative of $\sqrt{x}$ at $x = 9$ is $1/6$ which you can check by hand.

L'Hopital's Rule

Informally, L'Hopital's Rule says the following: consider two differentiable functions $f(x)$ and $g(x)$. Then

$$\lim \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \text{ OR } \frac{0}{0} \Longrightarrow \lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

Note that

$$\frac{\infty}{0} = \pm\infty \quad \frac{0}{\infty} = 0 \text{ are NOT admissible for L'Hopital's Rule!}$$

Integrability

Consider an interval $[a, b]$ and a partition $P_n = \{x_0, x_1, x_2, ...x_n\}$ with $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$.

If f is a function which lives on the interval, we will define the Riemann sum to be

$$S_n := \sum_{i=0}^{n-1} f(x_i^*)\Delta x_i$$

Here $x_i^* \in [x_i, x_{i+1}]$ is any point in the $i - th$ subinterval and $\Delta x_i = x_{i+1} - x_i$ is the subinterval width.

We want an upper and lower bound for these Riemann sums. For each subinterval $[x_i, x_{i+1}]$ of the partition P_n, we define M_i to be the supremum of f on the subinterval, and m_i to be the infimum of f on the subinterval. That is,

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$$

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

Now we define the Upper Riemann sum to be

$$U_n := \sum_{i=0}^{n-1} M_i \Delta x_i$$

and the Lower Riemann sum to be

$$L_n := \sum_{i=1}^{n-1} m_i \Delta x_i$$

Now, we choose a sequence of finer and finer partitions $\{P_n\}$ such that for all i, the subinterval width $\Delta x_i \to 0$ as $n \to \infty$. Note that we always have

$$L_n \leq S_n \leq U_n$$

So if *for any choice* of partitions P_n with $\Delta x_i \to 0$ for all i as $n \to \infty$, we need the following limits to be equal

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n \qquad (**)$$

If this key condition holds, by the squeeze theorem $\lim_{n\to\infty} S_n$ is well defined and we have

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i^*)(x_{i+1} - x_i)$$

In this case, f is said to be integrable on the interval $[a, b]$.

NOTE that $(**)$ must hold for ANY choice of finer and finer partitions in order for the integral to be well defined. However, there is a key **theorem** which tells us that we only need to find one sequence of partitions:

Theorem. If there exists any sequence of finer and finer partitions P_n (finer meaning $\Delta x_i \to 0$ as $n \to \infty$ for all i) such that $(**)$ holds, then the Riemann integral exists.

Integrability Theorems

Theorem. Any continuous function on a closed bounded interval is integrable.

Theorem. If f is a bounded function on a closed bounded interval $[a, b]$ and continuous except at finitely many points of $[a, b]$, then f is integrable on $[a, b]$.

Theorem. If f is a monotone function on a closed bounded interval $[a, b]$, then f is integrable.

Linearity of the Integral

If f, g are integrable functions and $\alpha, \beta \in \mathbb{R}$ are constants, then

$$\int \alpha f(x) + \beta g(x) dx = \alpha \int f(x) dx + \beta \int g(x) dx$$

That is, the integral is a linear operator. Also, if f, g satisfy $f \leq g$, then $\int f(x)dx \leq \int g(x)dx$ so the integral preserves order.

Let f be integrable on $[a, b]$, and set $M = \sup_{[a,b]} f(x)$ and $m = \inf_{[a,b]} f(x)$. Then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Note the important triangle inequality:

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

Finally, we need not require $a < b$ after defining:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Fundamental Theorems of Calculus

Theorem 1. Let f be a differentiable function on $[a, b]$ such that f' is integrable on $[a, b]$. Then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Theorem 2. Let $f(x)$ be integrable on $[a, b]$. Fix $c \in [a, b]$ and define

$$F(x) = \int_c^x f(x)dx$$

Then F is continuous on $[a, b]$, and at each point x where $f(x)$ is continuous, $F(x)$ is differentiable and

$$F'(x) = f(x)$$

Integration by Parts

Let f, g be differentiable on $[a, b]$ with integrable derivatives. Then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)$$

Improper Integrals

Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

and similarly

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

If a function is continuous but unbounded on $(a, b]$, (so it blows up at a), we can define the integral to be

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)dx$$

Example.

$$\int_0^1 \frac{1}{\sqrt{x}} = \lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{1}{\sqrt{x}}$$

Similarly, if the function is continuous but unbounded on $[a, b)$, (so it blows up at b) we define

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)dx$$

Series

Consider a sequence $\{a_n\}$ and now consider the sequence of partial sums

$$\{S_n\} = \{\sum_{i=1}^{n} a_i\}$$

If $\{S_n\}$ converges, we define

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} S_n$$

and we say that the series on the left hand side converges. If the sequence of partial sums $\{S_n\}$ fails to converge, we say that the series diverges.

n-th Term Test for Divergence

If $\{a_n\} \not\to 0$, then the series diverges.

p-Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff p > 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges } \iff p \le 1$$

Geometric Series Test

$$\sum_{n=0}^{\infty} Cr^n \text{ converges to } \frac{C}{1-r} \iff |r| < 1$$

Note that here the series starts at $n = 0$, which is just indicating that C is the first term of the sequence. For example,

$$\sum_{n=5}^{\infty} \frac{3}{5^n} = \frac{3/5^5}{1 - 1/5}$$

since $3/5^5$ is the first term of the geometric series with common ratio $r = 1/5$.

Comparison Test

If $|a_n| \le b_n$, then $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges. Also, $\sum a_n$ diverges $\Longrightarrow \sum b_n$ diverges.

Limit Comparison Test

If $0 < \lim \left|\frac{a_n}{b_n}\right| < \infty$, then $\sum a_n$ and $\sum b_n$ will converge or diverge together.

Ratio Test

If

$$\lim \left| \frac{a_{n+1}}{a_n} \right| < 1$$

then the series converges. If the above limit is > 1 then the series diverges. If the above limit is $= 1$ then we cannot conclude anything.

Alternating Series Text

An alternating series is of the form $\sum (-1)^n a_n$ where $a_n > 0$. The terms of the *sequence* we are summing over 'alternate' between positive and negative.

Conditional Convergence.

If $\lim |a_n| = 0$ then the alternating series converges conditionally.

Absolute Convergence.

If $\sum |a_n|$ converges then the alternating series converges absolutely (this is strongest).

Integral Test

If a sequence $\{a_n\}$ is positive and monotone decreasing, and we find a positive, monotone decreasing function f such that $f(n) = a_n$ for all n, then the series $\sum a_n$ will converge or diverge with the integral $\int_1^\infty f(x)dx$.

Finally, note that a series $\sum_{n=1}^\infty a_n$ converges if and only if its tail converges, that is, for any choice of N, $\sum_{n=N}^\infty a_n$ converges. Therefore if you don't like the first few terms of a series, just chop them off! You will affect the value of the summation, but not the convergence.

Series of Functions

Weierstrass M-Test

A series of functions $\sum_{k=1}^\infty f_k(x)$ on an interval I converges uniformly on I if there is a convergent series of positive terms

$$\sum_{k=1}^\infty M_k < \infty$$

such that $|f_k(x)| \leq M_k$ for all $x \in I$ and all k.

Example (Comparison Test Trick): Consider the series

$$\sum_{n=1}^\infty \frac{2n+1}{n^4 - 3n^2 + 2}$$

We can show convergence by limit comparison with $1/n^3$, and this is the best argument. However, to produce a direct comparison, we choose

$$\frac{2n+1}{n^4 - 3n^2 + 2} \leq \frac{2n+n}{n^4 - \frac{1}{2}n^4 + 0} \qquad \text{whenever } 3n^2 < \frac{1}{2}n^4$$

Lets examine the above inequality. To make a fraction bigger, I can make the numerator bigger and/or the denominator smaller. Replacing 1 with n in the numerator makes the fraction bigger. Then I replace 2 with 0 in the denominator to make the denominator smaller. Finally, I have to require $3n^2 < \frac{1}{2}n^4$ to replace the $-3n^2$ with a $-\frac{1}{2}n^4$ to make the denominator smaller.

Requiring $3n^2 < \frac{1}{2}n^4$ is equivalent to $6 < n^2$, so $3 \le n$. Thus we have

$$\frac{2n+1}{n^4 - 3n^2 + 2} \le \frac{2n+n}{n^4 - \frac{1}{2}n^4 + 0} = \frac{3n}{\frac{1}{2}n^4} = \frac{6}{n^3} \quad \text{whenever } n \ge 3$$

This trick produces a direct comparison with $6/n^3$, and $\sum 6/n^3$ converges by p-series test with $p = 3$.