MATHSHEETZ.COM

POLYNOMIALS

Contents

What is a Polynomial?

A polynomial is a type of function. A function is a rule that takes in a number x, called the input, and gives you back $f(x)$, the output. So exciting.

Polynomials are special functions that only have powers of x multiplied by numbers called "coefficients". For example,

$$f(x) = ax^2 + bx + c$$

is a degree 2 polynomial with cooefficients a, b, c. The **degree** of a polynomial is the highest power of x that shows up in the polynomial. The degree is also sometimes called the **order** of a polynomial.

Line Equation

A line is a degree 1 polynomial that takes the form

$$f(x) = mx + b$$

Sometimes we write $y = mx + b$ instead. To avoid confusion, always think about y and $f(x)$ as the same thing. Here m is the **slope** of the line, and b is the **y-intercept**.

The **slope** of a line measures how much y increases when x increases by 1. This is why slope is sometimes called "rise over run", and can also be thought of as the steepness of a line.

The **y-intercept** of the line is the point on the y-axis that is hit by the line. There is always exactly one point. To find the y-intercept, plug in $x = 0$ into the equation of the line.

Example 1. Find the slope and y-intercept of the line $y = 5x + 4$.

Solution. The slope is 5, which I read off as the coefficient (or number in front of) x. The y-intercept is found by plugging in $x = 0$ into the equation, and I get $y = 5 * 0 + 4 = 4$. Written as a point, the y-intercept is $(0, 4)$.

Graphing the Line

To graph the line, we need only to find two points. We can plug in any number we want in for x, and the function will give us an output y. For instance, using Example 1, I plug in $x = 1$ and get back $y = 5 * 1 + 4 = 9$. Now I have a point $(1, 9)$. If I know two points on the line, say $(0, 4)$ and $(1, 9)$, the line which goes through both of them is exactly what I want. See figure 1.

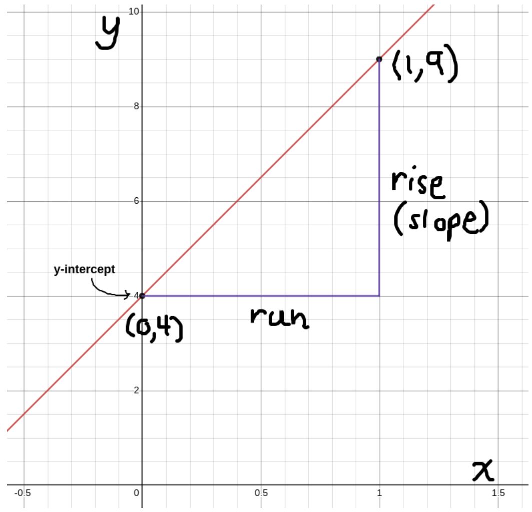


Figure 1.

Quadratics

A polynomial of degree 2 can be written in the form

$$y = ax^2 + bx + c$$

The **roots** of this polynomial, also known as the **x-intercepts**, are the values of x for which $y = 0$. To remember this, think: "y must be zero for the graph to hit the x-axis". For any quadratic polynomial, there is a formula for the roots, called the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (\star)$$

Sometimes the expression $b^2 - 4ac$ is negative, in which case the roots are imaginary numbers.

Example 2. Find the roots of $y = x^2 + 7$, and $y = 2x^2 - 4x + 2$.
Solution. Use the quadratic formula. In the first case, $a = 1, b = 0, c = 7$. Plugging these in to $(\star)$, we have

$$x = \frac{0 \pm \sqrt{0 - 4 * 1 * 7}}{2 * 1} = \frac{\pm\sqrt{-28}}{2} = \frac{\pm i\sqrt{28}}{2} = \frac{\pm i\sqrt{4} * \sqrt{7}}{2} = \pm i\sqrt{7}$$

For the second case, $a = 2, b = -4, c = 2$. The quadratic formula yields

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 * 2 * 2}}{2 * 2} = \frac{4 \pm \sqrt{0}}{4} = 1$$

Suspicious! There is only one root here. Quadratic equations always have two roots, whether real or imaginary. That must mean that our equation has the root $x = 1$, and the root $x = 1$. That is what we call a **multiplicity** of 2. Mathematically,

$$y = 2(x-1)^2$$

In factored form, the polynomial has the root $x = 1$, which corresponds to the factor $(x-1)$, showing up twice as $(x-1)^2$. This explains why a root seemed to disappear: it showed up twice as the same number.

Factoring

Factoring a polynomial means to rewrite the polynomial in terms of its roots. We use the roots as building blocks of the polynomial. For example, if I know that I have a degree 2 polynomial with **leading coefficient** of 4, and roots at $x = 2$ and $x = 7$, there is only one possibility:

$$y = 4(x-2)(x-7)$$

The leading coefficient is the coefficient of the highest order term (the x to the biggest power). Also, the roots $x = 2$ and $x = 7$ are turned into the building blocks $(x-2)$ and $(x-7)$. These blocks must go to zero exactly at the root, i.e., when I plug $x = 2$ into $(x-2) \to (2-2)$ I get zero.

This suggests a **method for factoring:**

if I give you a polynomial $ax^2 + bx + c$, use the quadratic formula to find the roots, say r_1, r_2. Then the factored form is always $a(x-r_1)(x-r_2)$. Dont forget a, the leading coefficient. Forgetting this number will make you lose an A in your class.

*** This method will always, always work.*** Even for higher order polynomials (though you dont have the quadratic formula). Yes, it works even if the roots are complex. You need only foil the complex building blocks and you will end up with a sensible equation. For example, foil $(x+4i)(x-4i)$ and you will get $x^2 + 16$.

There are other, faster ways to factor a quadratic. IF $a = 1$, and you just have $x^2 + bx + c$, you can ask the question: what q_1 and q_2 can I choose that multiply to get c, and add to get b? If you can guess these numbers, you can instantly factor the polynomial, but this time as $(x+q_1)(x+q_2)$.

For example, if I ask you to factor $x^2 + 7x + 12$, you might realize that $3 + 4 = 7$ and $3 * 4 = 12$. Then the factorization is $(x+3)(x+4)$. Done. *But the roots are NOT* $3, 4$. This is a common mistake. The roots make the building blocks go to zero. Therefore they are $-3, -4$ in this example.

Also, you can look to **complete the square:**

Completing the Square

Say I give you a polynomial $x^2 + bx + c$. Then ignore c (what a stupid letter), and simply use $\frac{b^2}{4}$ instead of c. Obviously you shouldn't be able to do that... right? Well, let's add zero to the equation, which is definitely allowed since zero is the nicest number. Except add $0 = \frac{b^2}{4} - \frac{b^2}{4}$ Mathematically,

$$x^2 + bx + c$$

$$= x^2 + bx + 0 + c \qquad \text{(adding zero)}$$

$$= x^2 + bx + \left(\frac{b^2}{4} - \frac{b^2}{4}\right) + c \qquad \text{(rewrite zero smartly)}$$

$$= \left(x^2 + bx + \frac{b^2}{4}\right) - \frac{b^2}{4} + c \qquad \text{(move parenthesis)}$$

Why did I go to all of this work? This expression seems even more gross and you are probably about to recycle this mathsheetz and return to your life. But it turns out, $x^2 + bx + b^2/4$ is actually a very friendly quadratic, in that it is a perfect square. Thus I "complete the square" as

$$= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c \qquad \text{(complete the square)}$$

Whew! Lets do an example.

Example 3. Complete the square on the polynomial $p(x) = 3x^2 + 6x + 21$.

I am not scared by the fact that this polynomial is called $p(x)$, not $f(x)$. I am also not scared by the leading coefficient of 3, although it is annoying. Lets start by bringing it out

$$p(x) = 3\left(x^2 + 2x + 7\right)$$

Now I have a quadratic building block $x^2 + 2x + 7$. I need 7 to be 1 to have a perfect square, so write

$$p(x) = 3\left(x^2 + 2x + 1 - 1 + 7\right)$$

Then I get a nice square and simplify:

$$p(x) = 3\left((x+1)^2 + 6\right) = 3(x+1)^2 + 18$$

In general, always bring out the leading coefficient first before completing the square. Even if it doesn't factor out nicely. For example,

$$5x^2 + 6x + 7 = \frac{5}{5}(5x^2 + 6x + 7) = 5\left(x^2 + \frac{6}{5}x + \frac{7}{5}\right)$$

Meditate on this expression silently for the next 30 minutes.
After your meditation, complete the square on $ax^2 + bx + c$. You will get

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}$$

and now you are officially considered a nerd for two reasons. 1. You are still reading this mathsheetz. 2. You now know how to prove the quadratic formula. Simply set the above equation equal to 0 and solve for x. Thats right, completing the square is *where the quadratic formula came from*.

Graphing Quadratics

There are a few ways to graph a quadratic.

Method 1: Find the roots. The vertex of the parabola is always halfway between the roots. Look at the sign of the x^2 term (leading coefficient). If the sign is positive, smiley face. If negative, frowny face.

Method 2: complete the square. Then you will have something that looks like

$$d(x+e)^2 + f$$

If those letters are annoying, you should get used to not caring. Its not good to be too dependent on a, b, c, after all, they are just letters and dont deserve special treatment for being the firstborn of the alphabet. In this completed square form, $x = -e$, $y = f$ will be the vertex. In general, *the vertex point makes the completed square go to zero*.

Combining method 1 and 2 should give you a pretty good picture. Here is an example:

Example 3.5: Graph the function

$$x^2 + 2x - 15$$

Solution. I notice $5 * (-3) = -15$ and $5 + (-3) = 2$. Aha, my polynomial is $(x+5)(x-3)$. So my roots are $-5, 3$. My vertex must be halfway between them, at $x = -1$. Plugging in $x = -1$, I get $(-1)^2 + 2(-1) - 15 = -16 = y$. Notice how y is always the value of the function (the output). So my vertex is $(-1, -16)$. Also, x^2 has a coefficient of 1 which is positive, so happy face. Done.

Higher Order Polynomials

Recall that the order of a polynomial is highest power of x that shows up. The important thing to know is that all higher order polynomials are built from simple building blocks. That is, if you give me ANY polynomial of the form

$$ax^n + bx^{n-1} + ... + cx^2 + dx + e$$

it can always be rewritten as

$$a(x - r_1)(x - r_2)...(x - r_n)$$

where $r_1, r_2, ..., r_n$ are the roots, and there are always exactly n of them (since n is the degree of the polynomial). But be careful, because I never said r_1 and r_2 had to be different numbers. The same root can show up more than once, and then however many times it shows up, that is its multiplicity. For example, the polynomial

$$4(x-3)^7(x-4)(x+2)^2$$

has a leading coefficient of 4, a degree of $7 + 1 + 2 = 10$, and roots at $3, 4, -2$. The root at $x = 3$ has a multiplicity of 7.

Also, please keep in mind that the roots $r_1, ..., r_n$ could be imaginary numbers. We just know that they definitely exist, and there are n of them. In fact, this is why the imaginary numbers were invented in the first place (so we could imagine the roots of a polynomial to exist even when they dont, so now they always do :).)

Polynomials of higher order can be factored using long division. That is our next big discussion.

Polynomial Long Division

To do polynomial long division, you are going to use the normal long division algorithm that is taught in most high schools (called the Euclidean division algorithm), and you will follow the exact same steps as you always have, *except* you will use the **leading term** (biggest power of x) of the polynomial in place of just a number. Lets recall some long division (yayyY)

```
   5906
7)41342
  35
   63
   63
    04
    00
     42
     42
      0
```

If that last number was not 0, but say like 2, then I would have a remainder of 2 and would have to add 2/7 to my final answer.
We are going to do the same thing, but now with polynomials, using the !! leading term of the divisor.

Example 4. Simplify

$$\frac{7x^5 + 3x^3 + 4x^2 + 22}{x^2 + 2x + 7}$$

I am going to write out my polynomial division, and I am going to put zeros in places where the degree is missing.

See how there is no x^4 term on the top? I am going to replace that with $0x^4$. Here. we. go.

$$
\begin{array}{r|rrrrrr}
 & 7x^3 & & & & & \\
\cline{2-7}
x^2+2x+7 & 7x^5 + & 0x^4 + & 3x^3 + & 4x^2 + & 0x + & 22 \\
 & 7x^5 + & 14x^4 + & 49x^3 & & & \\
\cline{2-4}
 & & -14x^4 - & 46x^3 & & &
\end{array}
$$

In this step I look at $7x^5$ and I take my leading term, x^2, and divide it into $7x^5$. I get exactly $7x^3$. Then I multiply the whole divisor polynomial (x^2+2x+7) by $7x^3$ and **subtract** it from my dividend polynomial (the bigger one in the flat house). Please remember that, although it is not written, I am subtracting $7x^5+14x^4+49x^3$ from $7x^5+0x^4+3x^3$. I will always be subtracting like this in the example. Write minus signs if it helps ya.

$$
\begin{array}{r|rrrrrr}
 & 7x^3 - & 14x^2 & & & & \\
\cline{2-7}
x^2+2x+7 & 7x^5 + & 0x^4 + & 3x^3 + & 4x^2 + & 0x + & 22 \\
 & 7x^5 + & 14x^4 + & 49x^3 & & & \\
\cline{2-4}
 & & -14x^4 - & 46x^3 & & & \\
 & & -14x^4 - & 28x^3 - & 98x^2 & & \\
\cline{3-5}
 & & & -18x^3 + & 102x^2 & &
\end{array}
$$

$$
\begin{array}{r|rrrrrr}
 & 7x^3 - & 14x^2 - & 18x & & & \\
\cline{2-7}
x^2+2x+7 & 7x^5 + & 0x^4 + & 3x^3 + & 4x^2 + & 0x + & 22 \\
 & 7x^5 + & 14x^4 + & 49x^3 & & & \\
\cline{2-4}
 & & -14x^4 - & 46x^3 & & & \\
 & & -14x^4 - & 28x^3 - & 98x^2 & & \\
\cline{3-5}
 & & & -18x^3 + & 102x^2 & & \\
 & & & -18x^3 - & 36x^2 - & 126x & \\
\cline{4-6}
 & & & & 138x^2 + & 126x &
\end{array}
$$

$$
\begin{array}{r|rrrrrr}
 & 7x^3 - & 14x^2 - & 18x + & 138 & & \\
\cline{2-7}
x^2+2x+7 & 7x^5 + & 0x^4 + & 3x^3 + & 4x^2 + & 0x + & 22 \\
 & 7x^5 + & 14x^4 + & 49x^3 & & & \\
\cline{2-4}
 & & -14x^4 - & 46x^3 & & & \\
 & & -14x^4 - & 28x^3 - & 98x^2 & & \\
\cline{3-5}
 & & & -18x^3 + & 102x^2 & & \\
 & & & -18x^3 - & 36x^2 - & 126x & \\
\cline{4-6}
 & & & & 138x^2 + & 126x & \\
 & & & & 138x^2 + & 276x + & 966 \\
\cline{5-7}
 & & & & & -150x - & 944
\end{array}
$$

We are done. The remainder is $-150x-944$, and we write

$$\frac{7x^5+3x^3+4x^2+22}{x^2+2x+7} = 7x^3-14x^2-18x+138+\frac{-150x-944}{x^2+2x+7}$$

If this seems confusing, there are much better examples you can find online and staple to this mathsheetz. I just randomly chose some polynomials and it got ugly. Sorry bout that.

There is this interesting alternative to polynomial long division called **synthetic division**. Lets cover that now.

Just kidding. Dont do synthetic division. Never do synthetic divison. If your professor makes you do synthetic division, I am sorry.

Synthetic Division

If you really want to know how to do synthetic division, look it up. You will find someone much more enthusiastic about it than me. Ill do my best, though.

There is one important idea, called the **rational root theorem:**

If a polynomial has INTEGER coefficients, its roots, if they are rational, will be of the form

$$x = \pm\frac{p}{q}$$

where p is a divisor of the constant term (last term, without the x attached to it), and q is a divisor of the leading coefficient (in front of the most powerful x with the highest power). Again, the roots dont have to be rational. This theorem is basically just for the special case that the roots actually end up being rational.

For example, $4x^{923}+348x^3+25x+42$ has a leading coefficient of 4 and a constant term of 42, so $p = 1,2,3,6,7,42$ and $q = 1,2,4$ should be the possible p and q values. The funny part is, it is insanely unlikely that this gross polynomial has rational roots to begin with. But your professor will be more nice with their choices of polynomials (maybe).

If we use the rational root theorem, we can guess and check some roots. To do the "check" part of the guess and check, we take our guess, say $x = \frac{a}{b}$, and divide the block $(x-\frac{a}{b})$ into the polynomial we want to factor. If the remainder is zero, we find that our guess was right and we factor $(x-\frac{a}{b})$ out of the polynomial. You can do this using polynomial long division. If you find a cute shorthand of writing the polynomial long division, that isn't actually any different from just polynomial division except you are too lazy to write the letter "x", you have achieved synthetic division.

Rational Functions

Now we get to the most fascinating section of this mathsheetz. Indeed, whenever a student asks me about a rational function, I smile little inside. Quadratically. Haha. ha.

Rational functions are polynomial versions of fractions. That is, a rational function might look like

$$\frac{ax^3+dx^2+qx+b}{cx^5+hx^2+w}$$

These letters mean "there is a number here, but I havent told you that number". Okay, now we apply our polynomial tools to factor the top and bottom polynomials. When we treat them seperately like this and factor, we will end up with something like

$$\frac{a(x-r_1)(x-r_2)(x-r_3)}{c(x-q_1)(x-q_2)(x-q_3)(x-q_4)(x-q_5)}$$

The r's are the roots of the top, and the q's are the roots of the bottom. Also, never forget that leading coefficient! Here it shows up as a and c. Now, once in this form, I look for any cancellations. If there are any blocks (factors) on the top and bottom that are the same, I cancel them. The blocks that get cancelled are called **holes** and they should be excluded from the **domain**. More on that in a sec. After doing cancellations, I draw the following conclusions:

$x = r_1, r_2, r_3$ are the roots (roots of top polynomial)

$x = q_1, q_2, q_3, q_4, q_5$ are the vertical asymptotes (roots of bottom polynomial)

Horizontal Asymptote:
*If the fraction is top heavy (degree of top polynomial bigger than degree of bottom polynomial), there is no horizontal aymptote. There is only a slant asymptote, which we still get by taking only the most powerful terms on the top and bottom (including their coefficients) but ignoring every lower power term.
*If the fraction is bottom heavy, the horizontal asymptote is $y = 0$.
*If the fraction is balanced (equal degree of polynomial on top and bottom), then the horizontal asymptote is $y = \frac{a}{c}$! That is, the fraction of the leading (most powerful) coefficients of the top and bottom.

the **domain** is all x that I am allowed to feed my function. I feed it almost everything except when the bottom is zero. That is, *the domain is everything except the roots of the bottom polynomial before we did any cancellations with the top.*

the **range** of the polynomial is all the numbers it can output. How far up and down can it reach? This question is in general more complicated, and you should probably ask google if you need to.

Graphing Polynomials

To graph an arbitrary polynomial, I consider its roots and plot them immediately. Then I plot the vertical asymptotes, near which I know the polynomial will skyrocket to infinity. Next I plot the horizontal asymptote, if there is one. Finally, if a root has multiplicity 2, it will just touch the x axis and bounce off, much like the classic x^2 function (which is a smiley face and has multiplicity 2 root at the origin).

Lastly, I look at the regions in between my roots and vertical asymptotes. I know the function is zero at the roots, and infinity at the vertical asymptotes. In between them, I mostly only care about the sign. I can tell by guess and check (using the factored form is much easier) if the function is negative or positive in some region. Once I figure out where it is negative and positive in between the roots and asymptotes, I do my best to draw the curve. After all, this isn't art class. Also, just use desmos.com/graphing :}.

Final Example. Graph and analyze the function

$$f(x) = \frac{(x^4 - 3x^3 - 28x^2)(x-2)}{3(x^4 - 7x^3 + 14x^2 - 28x + 40)}$$

Solution. First step is to factor the top and bottom. Oh hey, thats nice, the numerator is already a little factored for us. I immediately see that on the numerator, I can pull out an x^2, so I do that and get the top to look like $x^2(x^2 - 3x - 28)(x-2)$. Now I do a little mental math and figure out that $-7 * 4 = -28$ and $-7 + 4 = -3$. So that middle term gets factored further, and we are left with $x^2(x-7)(x+4)(x-2)$. That looks better.

Now the denominator. This time things are gross. I need to guess and check some roots using the remainder theorem to try and factor. I should try $p = 1, 2, 4, 5, 8, 10, 20, 40$ and $q = 1$. I am ignoring the 3 out front (leading coefficient) because it is already nicely factored out of the building block. Notice the parenthesis. If things weren't that nice, I would have to factor it out by hand. Now, I guess and check the roots using a calculator (or synthetic division, or polynomial long division) and find that 5 and 2 are the roots. So I use polynomial long division and divide out $(x-2)$ and $(x-5)$ (which can be done one at a time), and I end up with x^2+4. Putting that together, the factored denominator is $3(x^2+4)(x-5)(x-2)$. I rewrite my initial fraction as:

$$f(x) = \frac{x^2(x-7)(x+4)(x-2)}{3(x^2+4)(x-5)(x-2)}$$

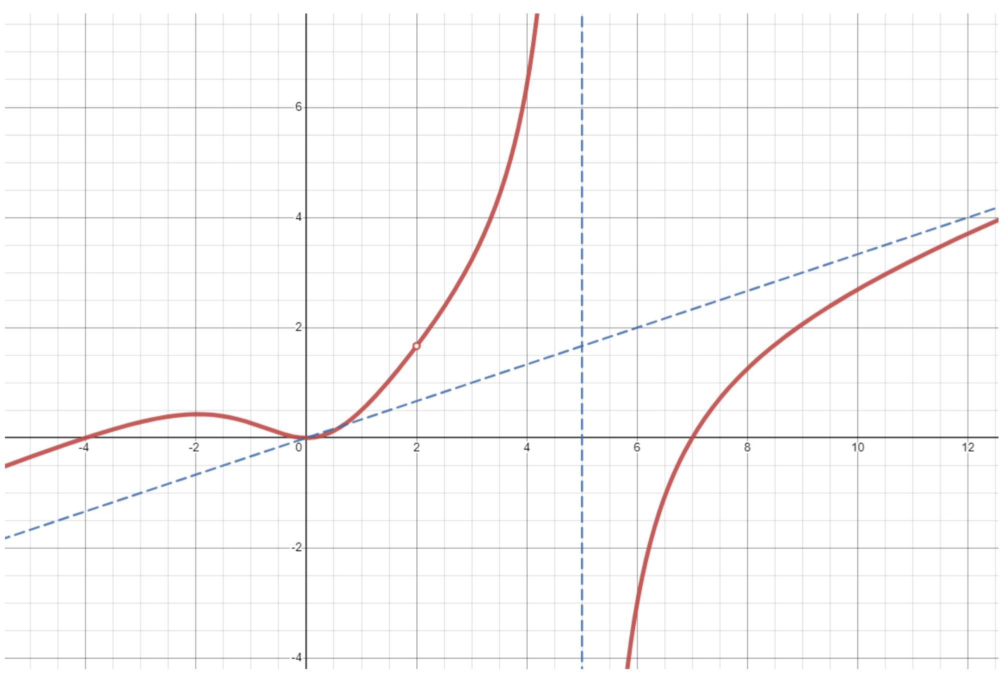
Okay, so immediately I see there is a cancellation about to happen with $(x-2)$. I make a note that $x-2=0 \iff x=2$ is a hole, and then I cancel the terms, leaving

$$f(x) = \frac{x^2(x-7)(x+4)}{3(x^2+4)(x-5)}$$

Great. Now I have roots at $x = 0, 7, -4$ because those are the roots of the top. Also, my $x = 0$ root is multiplicity 2 since the x is squared. The graph should just touch the x-axis at $x = 0$, not cross through it. All the other roots it will cross through. Looking at the denominator, we see that $x = 5$ is the vertical asymptote. There are no other vertical asymptotes since x^2+4 has imaginary roots. Also, the fraction is top heavy so there is no horizontal asymptote. If you care about the slant asymptote, I just take the most powerful terms on the top and bottom of the function and ignore everything else to find the slant asymptote. In this case, that looks like $x^5/3x^4 = x/3$, so the line $y = x/3$ is my slant asymptote.

I have roots $(0,0),(7,0),(-4,0)$ and immediately I plot them. I also plot the V.A. $x = 5$. Remember my hole at $x = 2$? Where should the hole be? Precisely where the function should have been if I was allowed to cancel. I plug in 2 to the simplified function (after cancellation) and that gives me the y-value of the hole. In this case it is $y = 4*(-5)*6/(3*8*-3) = 5/3$, and my hole is $(2, 5/3)$.

Now I look at the sign of the function. When x is less than -4, every simple (degree 1) block of the fraction is negative. That is, the blocks $(x-7),(x+4)$, and $(x-5)$ are all negative. Three negative signs is negative, so the graph goes to negative infinity far out on the left. As I hit the root at $x = -4$, the function crosses over the x axis and becomes positive. Then I touch the root at $x = 0$, go back up, and even shoot off to positive infinity at $x = 5$ (since it is a V.A). Crossing the V.A. at $x = 5$ I change sign, come down from negative infinity, and finally cross over one last time at the root $x = 7$ and head off to positive infinity. Whew. Done. The sign always flips at odd multiplicity roots and you cross over the x-axis, and the sign stays the same at even multiplicity roots and you touch the x-axis without crossing over. Same sign convention holds for the V.A. if the roots of the denominator polynomial have higher multiplicity. The graph looks like this:



Why do we care about Polynomials?

Lol. No comment.

Ill leave some space here for you to write a few sentences about why you care about polynomials and the essential role that they have played in your life journey. On the real, if you actually care about polynomials, or care about why people care about polynomials, check out the Stone-Weierstrass approximation theorem. Guess I did comment, after all.

Basically any smooth function can be approximated arbitrarily well by polynomials (its called the taylor series). Any mathematical or scientific model of the universe will likely have some polynomial, because they can approximate any continuous function to any degree of accuracy we want. Like compare the graph of $\sin(x)$ and

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$$

So ya. Polynomials are freaking important and show up everywhere in science and math. Think thats cool? Learn calculus. I even made a mathsheetz for it.

If you enjoyed this article, then check out my website mathsheetz.com for more content.