mathsheetz

CALC I

Limits

Calculus is, at its heart, the study of change. To study how a function $f(x)$ changes with x, we may first consider what happens to $f(x)$ as x approaches a certain value a. This is a **Limit**. Formally we will define

$$\lim_{x\to a} f(x) \equiv \text{the value } f(x) \text{ approaches as } x \text{ approaches } a$$

Now, this limit is only well defined if it is the same for *any* direction of approach. That is, if we approach a from the right ($x > a$), the function should approach the same value as when we approach a from the left ($x < a$). Mathematically, this idea is written as

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

If the function does not approach the same value in both directions, the limit does not exist. Figure 1 below demonstrates 3 critical examples. In the first 2 cases, the function *approaches* a single value from both sides. In the 3rd case, the right and left hand limits do not agree, and thus the limit does not exist (DNE).

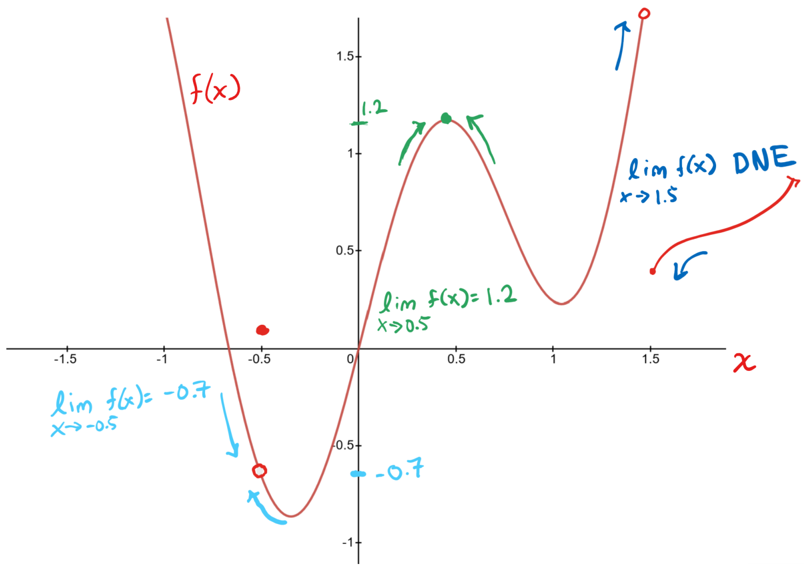


figure 1.

In practice, there are various tricks used to find limits. The most basic way to find a limit is to plug in the value of a into $f(x)$, which works for friendly, continuous functions. For example,

$$\lim_{x\to 5} x^3 = 5^3 = 125$$

We run into problems when $f(x)$ is not well defined at a. Most often the problem comes when a zero shows up in the denominator of f. To deal with this, we may need to factor terms and simplify, such as

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 2^2+2(2)+4 = 12$$

If the numerator does not go to zero at a, and the denominator does, then we will see the function explode to $\pm\infty$. The $\pm$ sign of infinity depends on the sign of the function itself. For example:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

$$\lim_{x\to 4^+} \frac{2x}{x-4} = \infty \qquad \lim_{x\to 4^-} \frac{2x}{x-4} = -\infty$$

Think about these examples. When $x > 4$, the fraction is positive, and when $x < 4$, the fraction is negative. Hence the difference in sign of ∞.

You may encounter less familiar functions, such as *piecewise* functions, in the study of limits. A piecewise function is just a function defined in seperate pieces. When evaluating the limit, consider each piece seperately. Example:

$$f(x) = \begin{cases} 2x & x \le 1 \\ \sqrt{x} + 5 & 1 < x \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} 2x = 0 \qquad \lim_{x \to 4} f(x) = \lim_{x \to 4} \sqrt{x} + 5 = 7$$

Try to show that the limit of $f(x)$ as $x \to 1$ does not exist. [Solution: the right hand limit is 6 (using $\sqrt{x}+5$ piece), and the left hand limit (using $2x$ piece) is 2. Since the right and left hand limits do not agree, the limit DNE.]

One of the most useful tools in evaluating a 0/0 limit is **L'Hopital's rule**, which states the following:

$$\text{if } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}, \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

In other words, if the limit of a fraction is zero/zero, you can take the derivative of the top over the derivative of the bottom and this fraction will give the same limit. **L'Hopitals rule is only true when the original limit is zero on the top and bottom of the fraction**.

Continuity

Continuity is the first, most basic measure of how 'smooth' a function is. Does the function abruptly jump at certain places? Or does it continue without breaks, like a curve that might be drawn with a single pencil stroke?

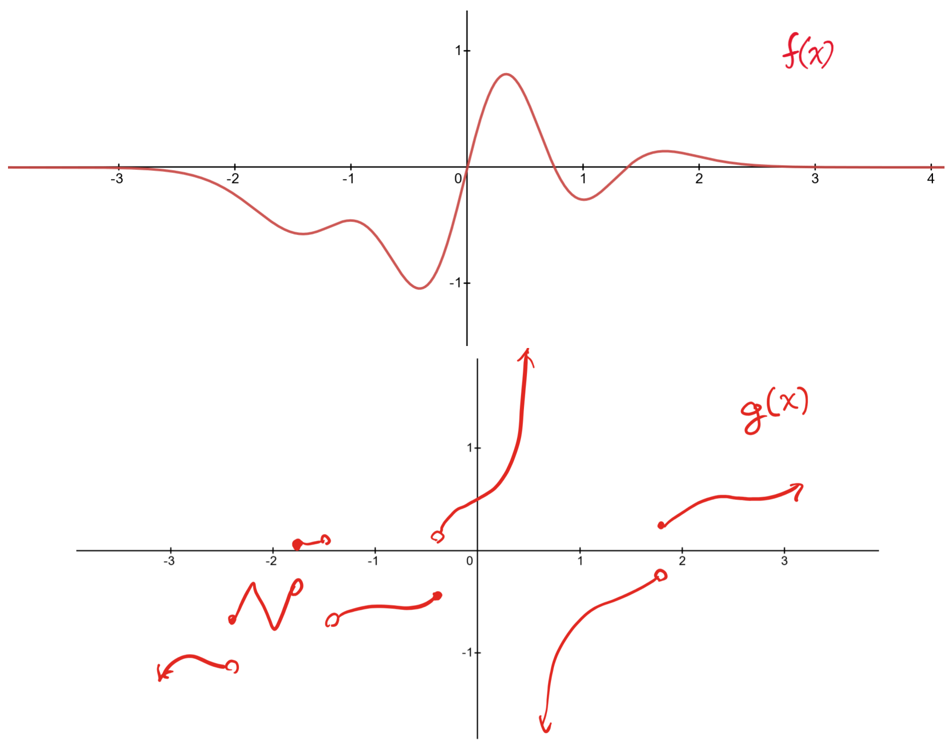


figure 2.

Figure 2 dramatically highlights the difference. Clearly $f(x)$ is continuous, and $g(x)$ is not. Notice $g(x)$ fails to be continuous due to 'jumps' at various places, and at a vertical asymptote (near $x = 0.5$).

Lets formally define a function to be **continuous at a point** $x = a$ when

$$\lim_{x \to a} f(x) = f(a)$$

A function is **continuous** if it satisfies this definition for every point. Notice that the left and right hand sides of the limit must agree, and this is what prevents the function from 'breaking' or 'jumping'. The only added condition is that the limit must also equal $f(a)$, the actual value of the function itself. This seems obvious, but it's not.
In figure 1, observe that the function is not continuous at $x = -0.5$ even though the limit is well defined, since

$$f(-0.5) = 0.1 \neq \lim_{x \to -0.5} f(x) = -0.7$$

This is an example of a point discontinuity. We can classify discontinuities into 3 types: jump, vertical asymptote, and point/removeable. We have seen them already. A **jump discontinuity** occurs when

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

A **vertical asymptote** (V.A.) occurs when

$$\lim_{x \to a^+} = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} = \pm\infty$$

Both of these discontinuities are shown in figure 2. Try to find them. [Solution: V.A. at $x = 0.5$ and many jumps, e.g. at $x = 1.8$].

Finally, a **point discontinuity** occurs when the limit exists, but is not equal to $f(a)$:

$$\lim_{x \to a} f(x) \neq f(a) \qquad\qquad \lim_{x \to a} f(x) \text{ exists, } \neq \infty$$

Point discontinuities are also called removeable discontinuities (e.g. fig 1. $x = -0.5$). Often they appear in a fraction of polynomials, when the denominator and numerator both go to zero, but can be factored so that these zero terms cancel (see the second example on limits).

Derivatives

Once we determine a function is continuous, we may ask the question: how quickly does it change? The answer:

The **derivative**, $f'(x)$, is the instantaneous rate of change of a function $f(x)$ at a point x.

There are two crucial concepts at work here: the *rate of change* of a function, and the notion of it being *instantaneous*. Both work in harmony to form the derivative. First, to measure the rate of change of $f(x)$ from point a to point b, we may want to use the expression

$$\frac{f(b)-f(a)}{b-a}$$

which is the slope of the line connecting $(a, f(a))$ and $(b, f(b))$. And indeed, this is a good approximation; in fact this is the **average rate of change** of $f(x)$ on the interval $[a, b]$. But there is a problem: we havent nailed down a single point in the interval. The function could fluctuate wildly between a and b, even if they are close together. Only when a and b become infinitesimally close, we can be sure that this is indeed the rate of change of $f(x)$ at the point a. And there you go: the derivative is instantaneous because it measures the rate of change at a single point, with infinitely small distance from a to b. Figure 3 illustrates this distinction.

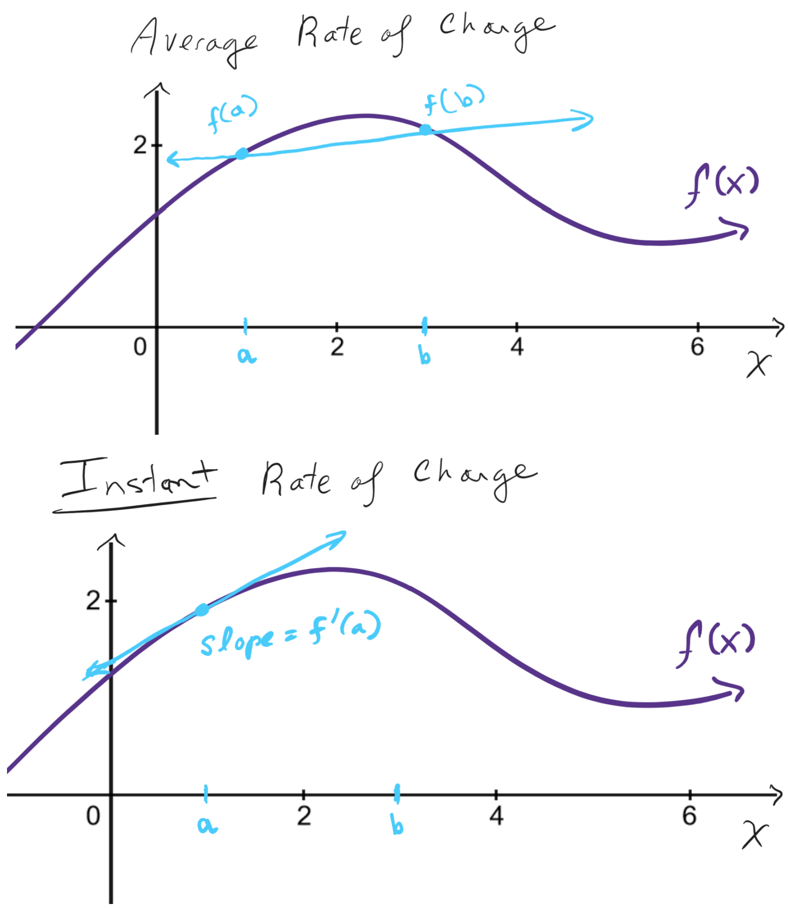


figure 3.

Now, define the derivative of f at a, $f'(a)$, to be

$$f'(a) = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}$$

If a function has a derivative at every point, it is called **differentiable**.

Note: The notation $\frac{df}{dx}$ is also used for the derivative of f with respect to x, and $\frac{d}{dx}$ for the derivative operation.

By using the limit, we bring b infinitely close to a, and now have a measure of the instantaneous rate of change of f at the point a.

The derivative may also be thought of as **the slope of the tangent line** of f at the point a. Informally, this is the slope of the line that perfectly aligns parallel with f at a. The tangent line is illustrated in figure 3 (bottom). Compare this to the *average* rate of change of f on the interval $[a,b]$, which is the slope of the **secant line** depicted in figure 3 (top).

A equivalent way to define the derivative (which is better for calculation) is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This is the exact same definition just using $a = x$, $b = x+h$. Using this definition, we can compute the derivatives of many common functions. A table of useful derivatives can be found anywhere on google and stapled to this mathsheetz (:

Now, there are fundamental rules concerning the derivative that can be proven using the above definition. If f and g are two differentiable functions, and c a constant real number, then:

a. The derivative of a sum is the sum of derivatives:

$$\frac{d}{dx}\left[f(x)+g(x)\right] = f'(x)+g'(x)$$

b. Constants (scalars) can be pulled out of the derivative:

$$\frac{d}{dx}\left(cf(x)\right) = cf'(x)$$

c. The derivative of a product of two functions is the derivative of the first times the second plus the derivative of the second times the first:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x)+f(x)g'(x) \qquad \text{(product rule)}$$

d. The derivative of a quotient of two functions is the derivative of the top times the bottom *minus* the derivative of the bottom times the top, all divided by the bottom squared:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2} \qquad \text{(quotient rule)}$$

e. The derivative of one function inside of another is the derivative of the outside times the derivative of the inside:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \qquad \text{(chain rule)}$$

Memorize these formulas using the mnemonic given after each lettering.

Implicit Differentiation

A traditional function is like a vending machine. It takes an input value x and gives you an output value y. We can write an **explicit** formula for this process, namely

$$y = f(x) \tag{$\star$}$$

which can also be written

$$y - f(x) = 0$$

However, sometimes what you want to eat is not in a vending machine. In this case, reaching the value y might be more complicated. For example, the equation

$$xy^2 + 3x\cos(\pi y) - x^3 = 14\sin(xy)$$

cannot be broken down into the basic $y = f(x)$ form. In this case, the formula relating x and y is called **implicit**. Clearly the implicit relationship between x and y, which I will call $g(x, y)$, is more complicated. In mathematical form, we write

$$g(x, y) = 0 \tag{$\star\star$}$$

In the simple case ($\star$), we used the derivative of y to find its instantaneous rate of change with respect to x, given by

$$\frac{dy}{dx} = f'(x)$$

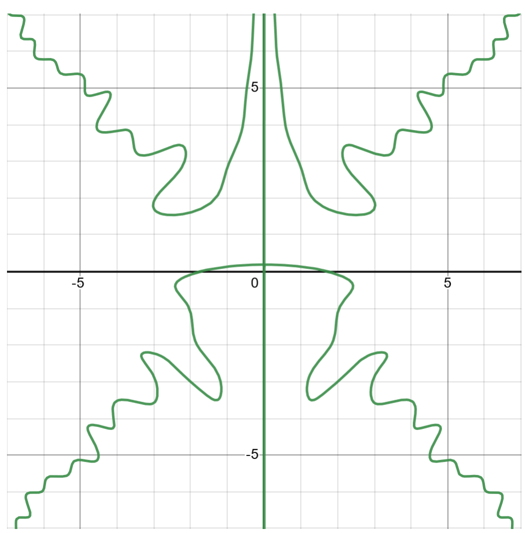
In the case of an implicit relationship, we can still find dy/dx. We need only invoke the chain rule on y. Therefore, we write $y = y(x)$ and differentiate ($\star\star$) as usual, with one catch: when x is differentiated it becomes 1, but every time y is differentiated it becomes dy/dx.

After differentiating, we will be left with an equation that contains x, y, and dy/dx, which we can manipulate to solve for dy/dx in terms of x and y. Let's see this in practice.

Example. A new cellular life form is discovered within the outer rings of Saturn. The organism uses a mixture of nitrogen and water molecules to perform a specific metabolic process called pathiation. The ratio between nitrogen (x) and water (y) molecules utilized depends on the developmental stage of the organism, but has been shown to miraculously satisfy the equation:

$$xy^2 + 3x\cos(\pi y) - x^3 = 14\sin(xy)$$

The solution curve of this equation is shown below, with standard axis (horizontal axis = x, vertical axis = y).



Determine the rate of change of water molecule consumption with respect to nitrogen uptake of the organism when their current levels of nitrogen and water molecules are 3.5 and 3.262 amol respectively.

Solution. We need to first decipher what the problem is asking us for. They want the "rate of change of water with respect to nitrogen". Water is defined as the y variable and nitrogen as the x variable, so they are just asking for the rate of change of y with respect to x - thats just the derivative $\frac{dy}{dx}$!

Now we turn to the equation given, and it looks intimidating. Recognizing that x and y are mixed together, we likely have an implicit equation, and therefore should take the derivative of both sides. The reader is encouraged to work this out on paper and compare their solutions, step by step.

$$xy^2 + 3x\cos(\pi y) - x^3 = 14\sin(xy)$$

First I will bring all terms over to the left side of the equation, mimicking $g(x, y) = 0$.

$$xy^2 + 3x\cos(\pi y) - x^3 - 14\sin(xy) = 0$$

Now I will differentiate the equation with respect to x. For visual aid, let $\Rightarrow$ denote taking the derivative. The first two terms get taken care of by the product rule, being very careful to treat every y using the chain rule.

$$xy^2 \;\Rightarrow\; (1)y^2 + x\left(2y\frac{dy}{dx}\right)$$

$$3x\cos(\pi y) \;\Rightarrow\; (3)\cos(\pi y) + 3x(-\pi\sin(\pi y))\left(\frac{dy}{dx}\right)$$

$$-x^3 \Rightarrow -3x^2$$

The most difficult term is last. For this case, we need to use the chain rule for $\sin(xy)$ and then the product rule on xy.

$$-14\sin(xy) \Rightarrow -14\cos(xy)\left(y + x\frac{dy}{dx}\right)$$

Putting everything together, we have

$$y^2 + 2xy\frac{dy}{dx} + 3\cos(\pi y) - 3\pi x\sin(\pi y)\frac{dy}{dx} - 3x^2 - 14\cos(xy)(y + x\frac{dy}{dx}) = 0$$

Next, we clean up the expression to isolate and solve for $\frac{dy}{dx}$. To do this, I will group terms that contain $\frac{dy}{dx}$ all on one side of the equation, factor $\frac{dy}{dx}$ out, and then divide by what was factored.

$$2xy\frac{dy}{dx} - 3\pi x\sin(\pi y)\frac{dy}{dx} - 14\cos(xy)(x\frac{dy}{dx}) = -y^2 - 3\cos(\pi y) + 3x^2 + 14y\cos(xy)$$

$$\frac{dy}{dx}(2xy - 3\pi x\sin(\pi y) - 14x\cos(xy)) = -y^2 - 3\cos(\pi y) + 3x^2 + 14y\cos(xy)$$

$$\frac{dy}{dx} = \frac{-y^2 - 3\cos(\pi y) + 3x^2 + 14y\cos(xy)}{2xy - 3\pi x\sin(\pi y) - 14x\cos(xy)}$$

Whew! And that is what we needed. Now I can plug in the initial information given by the problem, $(x, y) = (3.5, 3.262)$, to solve for the value of dy/dx that we care about. Using a calculator we find that

$$\left[\frac{dy}{dx}\right]_{(3.5,3.262)} = 1.660$$

Therefore, at the moment when its water and nitrogen levels are 3.262 and 3.5 amol, the organism consumes 1.66 amol of water per amol of nitrogen during pathiation.

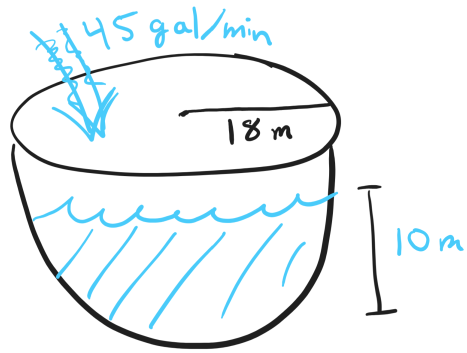
Related Rates

Many calculus problems involving implicit differentiation are given the infamous name 'related rates' because they ask questions about the derivative of one variable with respect to another. The basic way to solve these problems is as follows:

1) Identify what variable/derivatives are known, and which variable/derivative the problem asks you to solve for.
2) Create a general equation (without plugging in what you know) that relates these variables. Make sure the units are consistent.
3) Implicitly differentiate the equation by the right variable to get an equation involving the derivatives you need.
4) Finally, plug in what you know and solve.

The most important thing to remember is not to plug in information too quickly. First create a general equation, differentiate implicitly, and only then plug in the values given by the problem.

Example. Water flows into the top of a hemispherical bowl of radius 18 m at a rate of 45 gal/min. At what rate is the water level of the bowl increasing when it is 10 m above the bottom of the bowl?



Solution. Denote the water level from the bottom of the bowl by h, and the volume of the water by V. First, let us identify what we know. We know the radius of the bowl is 18 m, and that it is hemi-spherical. We also know that water enters the bowl at a rate $\frac{dV}{dt} = 45$ gal/min. Lastly, we know that the water level is currently 10 m.

The problem asks for the rate of change of h with respect to time, so we are trying to solve for $\frac{dh}{dt}$. Since we know $\frac{dV}{dt} = 45$ gal/min, we just need a formula that relates V, the volume of the water, to h.

Using volumes of rotation (discussed later in these notes), we can find the volume of the water at height h to be given by

$$V = \pi h^2 (18 - \frac{h}{3})$$

Where V is measured in m^3. We should keep in the back of our mind that eventually we will need to convert units between m^3 and gal, although not yet. First we should implicitly differentiate this equation. But with respect to what variable? h? Think about this. Since we need to solve for a time rate of change, $\frac{dh}{dt}$, we should differentiate both sides by t!

$$\frac{dV}{dt} = 36\pi h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt} = \frac{dh}{dt} \pi h (36 - h)$$

We have now reached the finish line. All that is left to do is *convert units*, 45 gal/min = 0.17 m^3/min, and plug in what we know, $\frac{dV}{dt} = 0.17$ m^3/min, $h = 10$ m. Thus

$$0.17 = \frac{dh}{dt} \pi (10)(36 - 10) \Rightarrow \boxed{\frac{dh}{dt} = 2.08 * 10^{-4} \text{ m/min}}$$

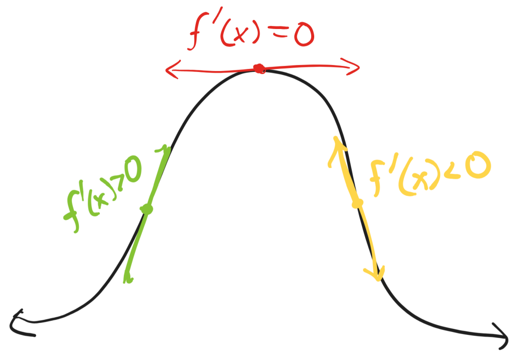
Max/Min and Optimization

Definition. An **extrema** of a function is a max or a min. Maxima and minima can be **local** or **global**. Local maxima are points at which the function is larger than its nearby values, whereas global maxima are points where the function is larger than its values everywhere. The same idea holds for minima.

Let's begin with a simple idea:

If a function $f(x)$ has a maximum (or minimum) at a point x_0, then $f'(x_0) = 0$ or DNE.

The sign of the derivative tells us if a function is increasing or decreasing. If $f'(x)$ is positive, then $f(x)$ is increasing with respect to x. If $f'(x)$ is negative, then $f(x)$ is decreasing with respect to x. At a maximum point, a function cannot be increasing or decreasing; hence the only possibility is $f'(x) = 0$ or DNE (does not exist). Visually,



Many calculus problems ask us to find the maximum or minimum of a function in a given interval. To do this, simply take the derivative of the function and set it equal to zero. All points x_0 such that $f'(x_0) = 0$ or $f'(x_0)$ DNE are called **critical points**.
Please note that the derivative is not defined at the endpoints of an interval $[a, b]$, so for any closed interval, $x = a$ and $x = b$ are always critical points.

Critical points are not always maxima or minima. Once we find the critical points, we must subject them to certain tests that tell us if they are actually extrema or not.

Test 0: Guess and Check (Brute Force). Once all critical points have been found, simply plug them all into the function and compare them to find the *global* maxima and minima.

Test 1: Sign Change of the Derivative. Suppose x_0 is a critical point. If $f'(x)$ changes sign across x_0 from positive to negative, then x_0 is a maxima (just like in the above figure). On the other hand, if $f'(x)$ changes sign across x_0 from negative to positive, then x_0 is a minima. If $f'(x)$ does not change sign across x_0, then x_0 is not an extrema.

Test 2: Sign of Second Derivative. Suppose x_0 is a critical point. If $f''(x)$ is negative, then x_0 is a maxima. If $f''(x)$ is positive, then x_0 is a minima. Graphically, a negative second derivative looks like a frowny face (creating a max), whereas a positive second derivative looks like a smiley face (creating a min).

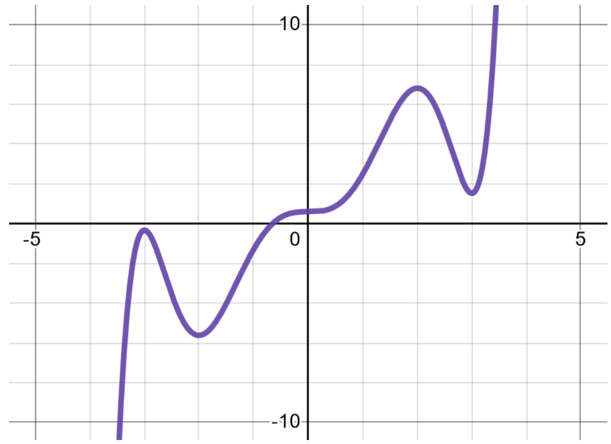
Let's see some optimization problems in action.

Example. A function $f(x)$ is given by

$$f(x) = \frac{5x^7 - 91x^5 + 420x^3 + 105}{175}$$

Perform an optimization analysis of $f(x)$ on the interval $[-5, 5]$. That is, determine any extrema of the function on the interval and classify them within the categories local/global and maxima/minima.

Solution. The graph of $f(x)$ is given below (using standard xy axis).



We begin by taking the derivative of f, and setting it equal to 0 at the critical points.

$$f'(x) = \frac{35x_c^6 - 455x_c^4 + 1260x_c^2}{175} = \frac{x_c^6 - 13x_c^4 + 36x_c^2}{5}$$

If x_c is a critical point of $f(x)$, then by definition

$$f'(x_c) = \frac{x_c^6 - 13x_c^4 + 36x_c^2}{5} = 0$$

Multiplying both sides by 5 and factoring by x_c^2 yields

$$x_c^2(x_c^4 - 13x_c^2 + 36) = 0$$

The right term can be further factored since $-4-9=-13$ and $(-4)(-9)=36$, giving us

$$x_c^2\left(x_c^2-4\right)\left(x_c^2-9\right) = 0$$

Therefore $x_c^2 = 0$, $x_c^2 = 4$ and $x_c^2 = 9$ all force the expression to zero, so $x_c = 0, \pm 2, \pm 3$ are the five critical points to consider, along with the endpoints of the interval $[-5,5]$.

If the problem only asks for global extrema, at this point we would simply apply Test 0 and be done. However, for a full analysis we also need to identify local extrema. Let's do the local analysis first, and then move to global. The factored form of $f'(x)$ will be extremely useful to us. We can even factor $f'(x)$ one step further, isolating all of the roots we just identified.

$$f'(x) = \frac{1}{5}x_c^2\left(x_c^2-4\right)\left(x_c^2-9\right) = \frac{1}{5}(x_c-3)(x_c-2)x_c^2(x_c+2)(x_c+3)$$

Now we conduct Test 1 by inspection. As x moves across $x_c=-3$, $f'(x)$ goes from positive to negative (since four negative terms (x_c-3), (x_c-2), (x_c+2), (x_c+3) become three negative terms (x_c-3), (x_c-2), (x_c+2)). Similarly, $f'(x)$ goes from negative to positive across $x_c=-2$, positive to negative across $x_c=2$, and negative to positive across $x_c=3$. By test 1, $x_c=-3,2$ are maxima and $x_c=-2,3$ are minima. Also, no sign change occurs across $x_c=0$ so it is not an extrema.

For the endpoints of the interval, since $f'(x)$ goes from positive to endpoint across $x_c=5$, it is a local maxima. Also, since $f'(x)$ goes from endpoint to positive across $x_c=-5$, it is a minima. Finally, we now plug in $x_c=\pm 2, \pm 3, \pm 5$ and apply Test 0 to determine by direct comparison that $x=-5$ is the global min, and $x=5$ is the global max on the interval. These conclusions are readily verified by the above graph.

Notice how the endpoint analysis was done analogous to any other critical point, replacing the word 'positive/negative' with 'endpoint' on the right for a right endpoint, and on the left for a left endpoint.

The Riemann Integral

The integral is perhaps the most useful and beautiful tool in all of mathematics. At its core, the integral is a way to add infinitely many small pieces together. Mathematically,

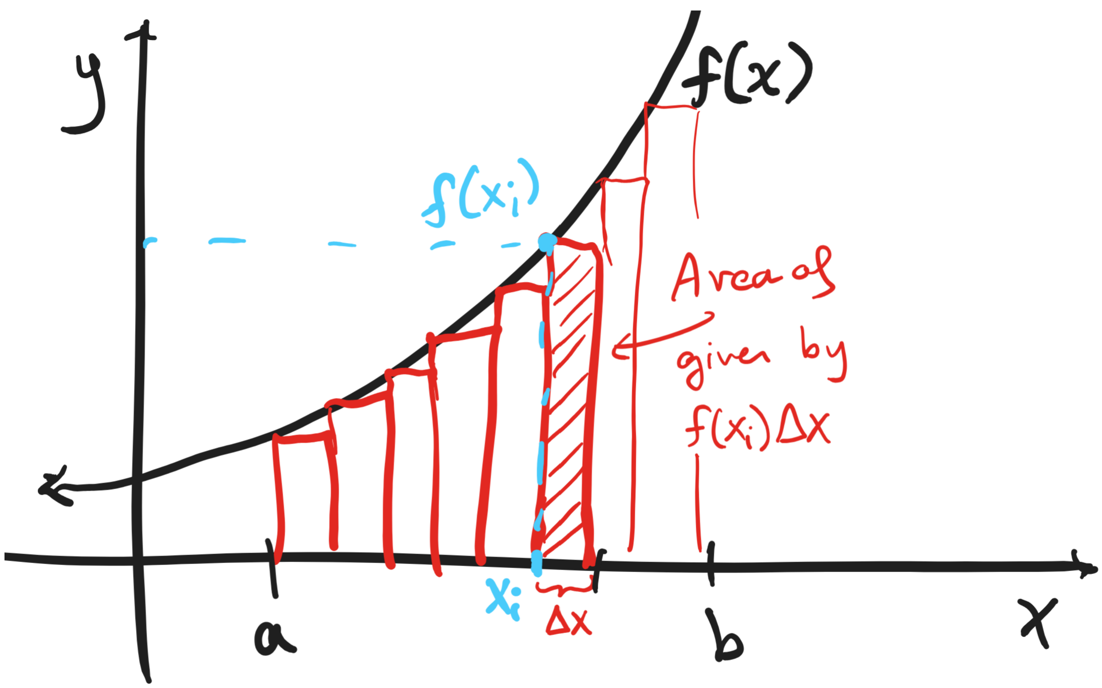
$$\int f(x)dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x$$

This definition is quite abstract. Here are a few insights: the Δx represents the size of the pieces we are adding together. Hence, $\lim_{\Delta x \to 0}$ just means the pieces are getting smaller and smaller, approaching size 0. Finally, the $f(x_i)$ represents a function that tells us what *the pieces are*, and the $\sum$ sign indicates that we are adding all of these pieces together.

We can make this definition more concrete by giving it a new meaning: the area underneath the curve. That is

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x = \textbf{the area underneath } f(x) \textbf{ on the interval } [a,b]$$

We can then view the Riemann integral as a mathematical construction to find this area through the following method. First, we construct boxes of width Δx and height $f(x_i)$ and arrange them underneath the curve $f(x)$ in an appropriate manner, shown below.



If we take the area of each rectangle, which is given by height × width = $f(x_i)\Delta x$, and we add all of these areas together, we get a good approximation of the area under the curve $f(x)$. This approximation is literally the summand $\sum_i f(x_i)\Delta x$. For a perfect approximation, we need only use more and more finely spaced boxes, thus sending their width to zero (hence the $\lim_{\Delta x \to 0}$).

Now, one brief detail must be mentioned: the width (Δx) need not be the same for every box, so long as all of the boxes are still getting infinitely small in the limit. Therefore, we could have written Δx_i instead of Δx to denote that the width of each box might not always be the same. If this detail is confusing to you, please dont be too concerned as long as you understand the above general construction.

There is a beautiful fact that relates the integral to the derivative: they are opposite to each other. That is, a derivative undoes an integral, and an integral undoes a derivative. This is called the **Fundamental Theorem of Calculus (FTC)**. Mathematically,

$$\frac{d}{dx}\int f(x)dx = f(x)$$

$$\int f'(x)dx = f(x)$$

If we take an integral of a function, and then take the derivative, we get back to the original function (and vice versa). Now, this means a lot for us when it comes to actually solving an integral. When faced with $\int f(x)dx$, we really just need to ask ourselves "what function, when I take its derivative, will give me $f(x)$?". This function is precisely the value of the integral, and is also called an **antiderivative**. You wouldnt be wrong to think that the antiderivative and the integral are the same thing!

Finally, if an integral is over a specific interval $[a,b]$, we should get a number that represents the area. To find this number, we need to first find the antiderivative $F(x)$, and then evaluate F as $F(b)-F(a)$. That is,

$$\int f(x)dx = F(x)$$

$$\int_a^b f(x)dx = F(b)-F(a)$$

Since the last step is just plugging in b and a into $F(x)$ and subtracting, the crucial step is really finding $F(x)$, the antiderivative. This takes practice, but there are a ton of general rules for finding the antiderivative that are the exact opposite of the derivative rules! A table of useful integrals and common integration rules can be found anywhere on google and also stapled to this mathsheetz (:

One crucial integral formula you must memorize is called the **Mean Value Theorem (MVT)**. There is a derivative and integral form of this theorem, and both versions are equivalent under the fundamental theorem of calculus. The MVT **derivative form** states that for a differentiable function $f(x)$ over the interval $[a,b]$, one can always find a value c within $[a,b]$ so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The MVT **integral form** states that for a continuous function $f(x)$ over the interval $[a,b]$, one can always find a value d within $[a,b]$ so that

$$f(d) = \frac{1}{b-a}\int_a^b f(x)dx$$

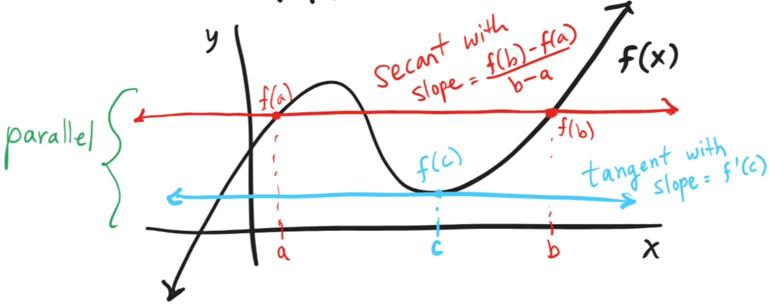
The c and d values guaranteed by the theorem need not be the same number. Lets spend some time interpreting what this theorem means.

Taking a look back at figure 3.(top), we see that in fact $\frac{f(b)-f(a)}{b-a}$ is just the slope of the secant line connecting $f(a)$ and $f(b)$. Therefore, the derivative mean value theorem says that there is some point c in the interval $[a,b]$ for which the function $f(x)$ has a tangent line with the same slope $f'(c)$ as the slope of this secant line. You could also say that at the point c, the function $f(x)$ has a tangent line *parallel* to this secant.

The integral form of the MVT states that there is some point d in the interval $[a,b]$ so that a box of width $b-a$ and height $f(d)$ has the same area as the total area under the curve $f(x)$. Make sure you understand both of these interpretations. I will provide a visual aid on the next page.

Try to use the remaining space of the next page for the most crucial **exercise** of this mathsheetz: I want you to figure out why the Fundamental Theorem of Calculus is true (you can research it if you want), and then explain why it is true beneath the MVT figure. If you are truly motivated, do the same for the Mean Value Theorem. These are unquestionably the most important theorems in Calculus I.
I will include my own solutions in Appendix A. If you look at these solutions before attempting to figure this out by yourself, please pay my website a 2 dollar fee (;.

MVT (derivative)

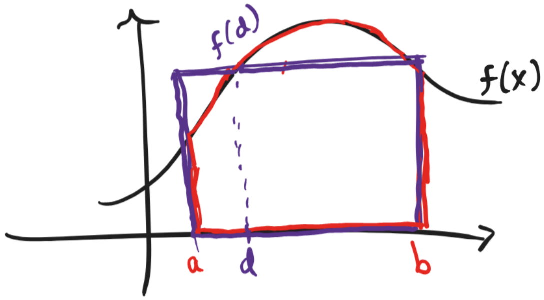


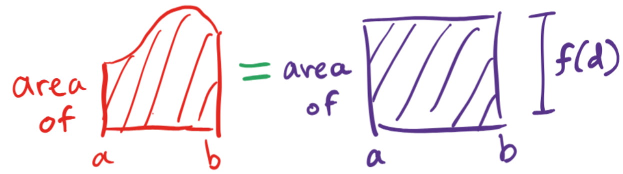
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(d) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Mean Value Theorem

MVT (integral)





Volumes of Rotation - Theory

Instead of reducing the integral to "the area under a curve", we shall return to the more general idea of an integral as adding up many small pieces of something.

If I take a function, rotate it around the x-axis, and ask you to find the volume of the object formed by the rotation, you will need to use an integral. This time, however, instead of uses small rectangles with areas $da = f(x)dx$, you will look for small disk/washer volumes dV and add up the volumes of these many disks.

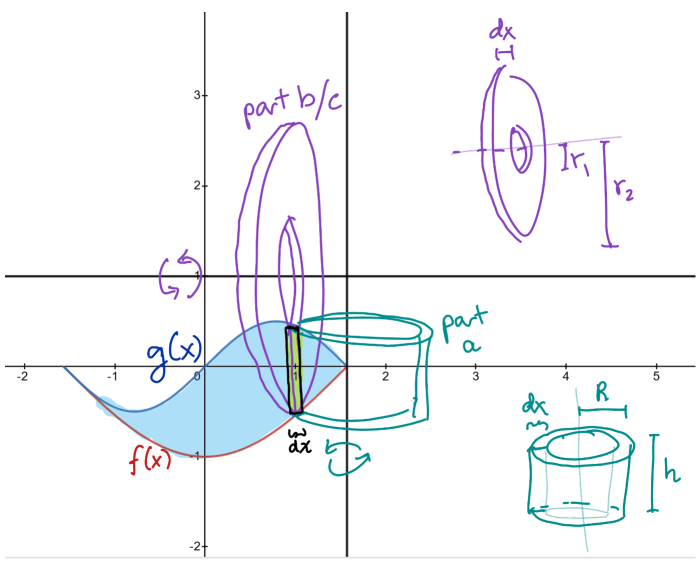
Therefore, the overall process is the same: first, you must determine the volume/area of the small pieces that you need to make the approximation. Then, you perform an integral over all of these pieces.

$$V = \int_a^b dV = \int_{\text{Domain}} (\text{Volume of Small Piece in Domain})$$

Getting good at doing volumes of rotation requires two skills: 1. an understanding of what the integral means, as explained above, and the pieces we need to use to approximate something. 2. practice and working examples. We used small rectangles of area $da = f(x) * dx$ to approximate the area under the curve. That is exactly why $\int f(x)dx = \int da = A$ is normally thought of as the area under the curve. For a different object, say a volume of rotation, all we need to do is figure out the volume dV of the pieces we need, and then the integral will fall into place.

Here is an **example**. Consider the region between $x = -\pi/2$ and $x = \pi/2$, bounded by $f(x) = -\cos(x)$ and $g(x) = \sin(x)\cos(x)$. Find the volume of rotation formed by rotating this region about: a) the line $x = \pi/2$, b) the line $y = 1$. Finally, c) find the volume of the hole in part b).

Solution. First, always draw a clear sketch of the rotation, and identify one of the key small volume pieces as the result of rotating our best friend: the usual rectangular box of width dx!



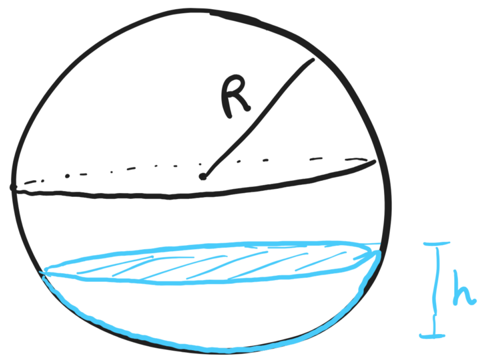
For part a), we find that the rectangular box (yellow highlight) becomes a cylindrical shell of volume $dV = 2\pi Rh * dx$. We can recognize $h = g(x) - f(x)$ as the height of the box, but R is a bit more tricky. The box is at horizontal location x, so the distance R is just the horizonal distance between the box and the line $x = \frac{\pi}{2}$, which is $R = \frac{\pi}{2} - x$. Putting this all together, we have $dV = 2\pi(\frac{\pi}{2} - x)(g(x) - f(x))dx$, and therefore, if we add all of these pieces up (each cylindrical shell, corresponding to a unique box running over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$), we obtain

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi\left(\frac{\pi}{2} - x\right)\left(\sin(x)\cos(x) - \cos(x)\right) dx$$

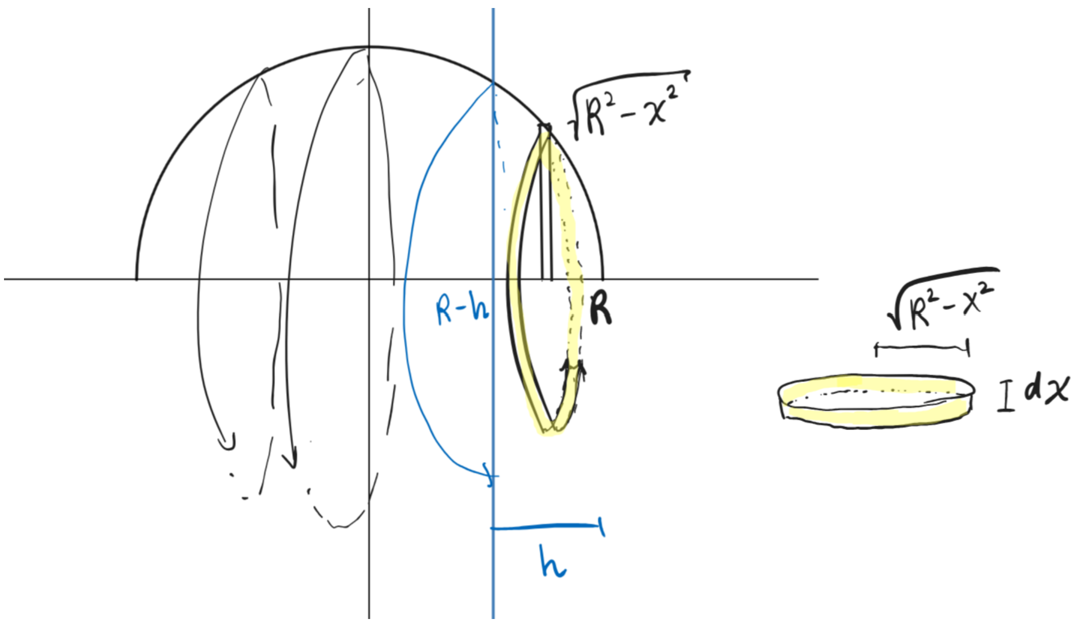
Now you try. Set up the integral for part b) and c). The integral will be almost the exact same, except now dV is the volume of the purple washers (and in c), their holes) instead of the volume of the green cylindrical shells.

Volumes of Rotation - Example

Example. Let S be a sphere of radius R. Find the volume inside of the bowl formed by slicing the sphere at a height h above its lowest point.



We observe by symmetry that the bowl is formed by rotating the curve $y = \sqrt{R^2 - x^2}$ about the x-axis, from $x = R - h$ to $x = R$. Visually, this rotation looks like



To find the volume of the bowl, we need to add up the volumes of each disk slice of width dx and radius $\sqrt{R^2 - x^2}$. These disks have volume

$$dV = \pi\left(\sqrt{R^2 - x^2}\right)^2 dx = \pi(R^2 - x^2)dx$$

The integral acts as a beautiful tool to sum the volumes of all disks, yielding

$$V = \int_{\text{disks}} dV = \int_{R-h}^{R} \pi(R^2 - x^2)dx$$

$$V = \pi\left[R^2x - \frac{1}{3}x^3\right]_{R-h}^{R} = \pi\left[\frac{2}{3}R^3 - R^2(R-h) + \frac{1}{3}(R-h)^3\right] = \boxed{\pi h^2(R - \frac{h}{3})}$$

Appendix A

Why the Fundamental Theorem of Calculus is true: It is fundamentally true because of calculus.

More specifically, if I consider the meaning of

$$\frac{d}{dx}\int f(x)dx$$

This expression reads "the instantaneous rate of change *of the area under the curve* of $f(x)$". If I start at a point x, and move slightly to the right, I need to add a thin box of height $f(x)$ to my area approximation. Therefore, the area increases at a rate exactly equal to $f(x)$, the current height of the function (and hence the approximation box at that point).

Why the Mean Value Theorem is true: On average, if we guess and check the theorem for enough functions, it seems to be true and is therefore true, based on the convergence of the sample mean to the truth.

Just kidding. The Mean Value Theorem (integral form) is true because, over a closed interval $[a,b]$, a continuous function always reaches a maximum and minimum value, say at x_{max} and x_{min} (as we learned in the optimization section!). Now, if we look at the boxes with width $(b-a)$ and height $f(x_{max})$ and $f(x_{min})$, we find that the area of the maximal box must be larger than the area under the curve, and the area of the minimal box must be smaller than the area under the curve. Thus

$$(b-a)*f(x_{min}) \leq \int_a^b f(x)dx \leq (b-a)*f(x_{max})$$

But now, the continuity of $f(x)$ ensures that there has to be some point d for which $(b-a)*f(d)$ is exactly equal to $\int_a^b f(x)dx$, since as x moves from x_{min} to x_{max}, the values of $f(x)$ increase smoothly along a continuum and therefore must, at some point d, perfectly hit any value in between $f(x_{min})$ and $f(x_{max})$.

Finally, the derivative version of the MVT can be seen by considering the integral version in the following manner:

$$f(d) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}(F(b)-F(a)) = \frac{F(b)-F(a)}{b-a}$$

But, by the fundamental theorem of calculus, $f(x) = F'(x)$, so $f(d) = F'(d)$ and we are left with

$$F'(d) = \frac{F(b)-F(a)}{b-a}$$

which is the exact statement of the MVT in derivative form. Remember, even though it is an antiderivative (and hence has a derivative $f(x)$), F is still any arbitrary function, which means that all jokes aside this is a perfectly valid way to prove the MVT.

I hope you enjoyed this mathsheetz! Right now, I am working to improve this free mathematical content. Please leave a review at mathsheetz.com, so I can improve these sheetz for students like yourself.